Nonlinear Wave Equations and Numerical Relativity

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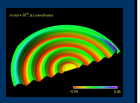
Modeling Gravitational Collapse:

- Difficulties:
  - Coordinates
  - Stability (form of equations)
  - Stability (approximation scheme)
  - Singularity formation
  - Computing resources
- Previous work in 1D and 2d:
  - 1D spherical symmetry
  - 2D axisymmetry



Choptuik,Hirschmann,Liebling,Pretorius, PRD 68,044007(2003)gr-qc/0305003
Choptuik,Hirschmann,Liebling,Pretorius, CQG 20, 1857 (2003)gr-qc/0301006

Lessons Learned:

- Tendency for "good" results to appear only above some minimum resolution
- Visualization and control over data important
- Small turn around time needed

It's hard!

The Move to 3D GR

- Lots to explore:
  - Singularity excision within horizon
  - Different formulations & differencing schemes
  - Interesting matter sources:
    - Fluids
    - E&M
    - MHD
- Get a handle on computing resources and visualization first… toy model in 3D

The Nonlinear Sigma Model

- Map from Minkowski (3+1) to target manifold (S^3)
- Symmetry breaking SO(4) → SO(3)
- Map satisfies: $\partial^\mu \partial_\mu \phi^A + \Gamma^A_{BC} \partial_\mu \phi^B \partial^\mu \phi^C = 0$
- NO gravity
- Choose hedgehog ansatz (spherical equivariant):

$$\phi^a = \begin{pmatrix} \sin\chi(r,t)\sin\theta\sin\varphi \\ \sin\chi(r,t)\sin\theta\cos\varphi \\ \sin\chi(r,t)\cos\theta \\ \cos\chi(r,t) \end{pmatrix} \qquad \ddot{\chi} - \frac{1}{r^2}\left(r^2\chi'\right)' = -\frac{\sin(2\chi)}{r^2}$$

- Already Known:

small initial data – no singularities "global existence"
large initial data – forms singularity

What happens between small and large?

- Parameterize initial data: $\chi(r,0) = Ae^{-(r-R_0)^2/\delta^2}$, $\dot{\chi}(r,0) = 0$
- Find:
  - A_{high} for which singularity forms
  - A_{low} for which energy disperses
- Tune for threshold A^*
  - Singularity formation for: $A > A^*$
  - Dispersal for: $A < A^*$

Critical Behavior

Liebling,Hirschmann,Isenberg, JMP 41,5691(2000)
Bizon,Chmaj,Tabor, Nonl.13,1411(2000)

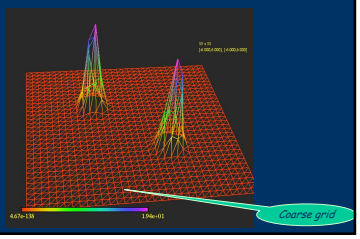
- Evolution approaches a self similar solution
- Critical solution is one of a family of solutions found by Aminneborg,Bergstrom, PLB 362,39(1995)astro-ph/9511064
- Critical solution sits on codimension 1 boundary between two basins of attraction
- Single unstable mode takes solution away from this boundary manifold

Move into 3 Spatial Dimensions

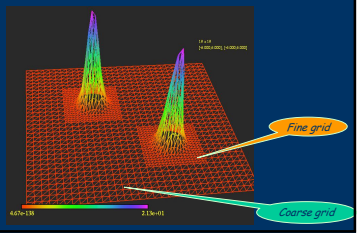
- Map satisfies: $\partial^\mu \partial_\mu \phi^A + \Gamma^A_{BC} \partial_\mu \phi^B \partial^\mu \phi^C = 0$
- 3D … pick Cartesian coordinates
- Pick generalized hedgehog: $\chi(r,t) \to \chi(x,y,z,t)$
- Nonlinear equation: $\ddot{\chi} = \chi_{,xx} + \chi_{,yy} + \chi_{,zz} - \frac{\sin 2\chi}{r^2}$
- Initial data: distorted Gaussian pulses, toroid, etc
- 3D is harder:
  - Tendency toward more fields
  - More memory/field 1000pts->8kB vs (1000)^3->1GB
  - More operations per point

need more computing power

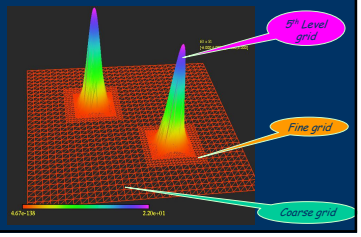
Adaptive Mesh Refinement



Adaptive Mesh Refinement



Adaptive Mesh Refinement



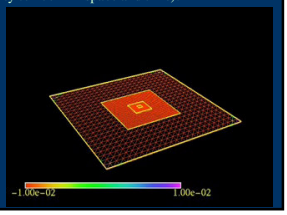
Adaptive Mesh Refinement (AMR)

- Dynamically add fine sub-grids… *mesh refinement*
- Finer resolution in time and space
- Fine grids created/destroyed as needed…*adaptive*
- Recursive…fine grids can have finer grids
- Computational work scales with resolution
- Grids communicate so that they fit together
- Resolution scales as a power law!
- Example: 15 levels of refinement (2:1)

$2^{15} \approx 33{,}000$

Resolving power

- Importance of visualization (ability to zoom in space and time):



Need More Resources

- Need quick turn around for testing/development
- Need lots of memory
  even w/ AMR, need to store grids someplace
- Never enough computing resources
- Jump in dim. D from 2 to 3 large:
  - CPU time scales: $O(N^{D+1})$
  - Memory scales: $O(N^{D})$

N – number of grid points

- Take advantage of trend towards cluster computing

Distributed Computing

- Network cheap PC "nodes"
  (fast ethernet, gigabit, myrinet)
- Build your own &/or use someone else's (e.g. NPACI)
- Need to communicate across nodes: Message Passing Interface (MPI)
- Non-trivial to distribute the problem across nodes
- Load-balancing can be tough, but don't need optimal…worst case. get good memory scaling

Distributed AMR

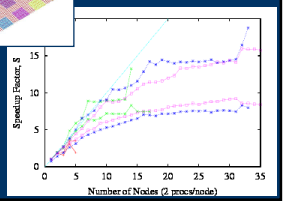
- AMR spread across many processors
- Strategy: distribute entire grids to different proc's
- Scheme:
  - master process orchestrates
  - slaves loop for commands
- Pro:
  - Straightforward to implement
  - Works well for large numbers of grids
- Con:
  - Not optimal, would like more autonomy
  - Horrible scaling for completely nested grids

--Same code can also do domain decomposition

--need for general grid based libraries

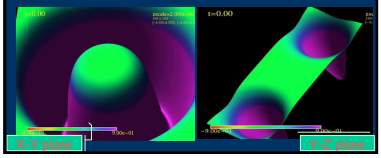
Early Distributed AMR Scaling

Speedup for N processors: $S = \frac{T_1}{T_N}$



Flatspace Critical Phenomena

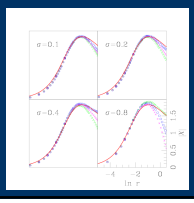
- Tune based on singularity formation
- Self-similar critical solution appears
- Toroidal Initial data



Results in 3D

Liebling,
PRD 66,041703(2002) gr-qc/0202093

- Same, spherically symmetric critical solution



Other Models

…work in progress

Other Models: Semilinear Wave Eq.

Bizon,Chmaj,Tabor, math-ph/0311019

- Simplest nonlinear wave equations
- Cartesian coordinates

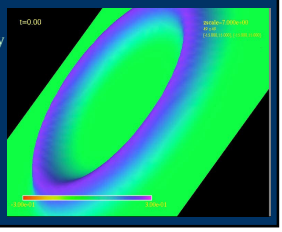
$$\ddot{\chi} = \chi_{,xx} + \chi_{,yy} + \chi_{,zz} + \chi^p$$

- Spherically symmetric results show "conventional" critical behavior for p=7

$$\chi^*(r,t) = (T-t)^{-1/3} U_1\left(\frac{r}{T-t}\right)$$

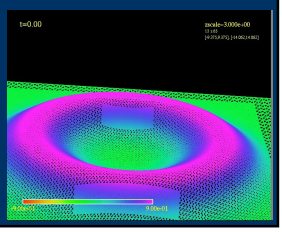
Other Models: Semilinear Wave Eq.

- Spherically symmetric initial data



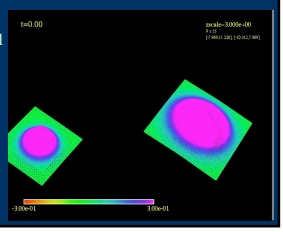
Other Models: Semilinear Wave Eq.

- Toroidal initial data



Other Models: Semilinear Wave Eq.

- Two initial pulses



Other Models: Maxwell Dilaton

Hirschmann,Liebling

- Action

$$S = \int d^4x \sqrt{-g}\left[R - a_0(\partial\phi)^2 - a_1 e^{-2\kappa\phi} F_{ab}F^{ab}\right]$$

- Choose flat space (R=0)
- Dilaton attractive… possible critical behavior
- Can't have spherically symmetric solutions
- Results not clear yet….looks like critical behavior though

Other Models: Ricci Flow on Sphere

- Critical behavior found in Ricci Flow

Garfinkle,Isenberg,math.dg/0306129

- One parameter families of geometries on S^3
  neck pinches…corseted spheres
- Two end states:
  - Round sphere
  - Unbounded curvature
- Critical solution: "javelin"
  …curvature growth at poles

Other Models: Ricci Flow on Sphere

Garfinkle,Isenberg,Liebling

- Idea: similar problem with less symmetry
- Need to maintain regularity at the poles
- Define geometry on S^2 using two patches:

Gomez,Lehner,Papadopoulos,Winicour
CQG 14,977(1997)gr-qc/9702002

- Stereographic projection of sphere: $(\theta,\varphi) \to (N_1, N_2)$

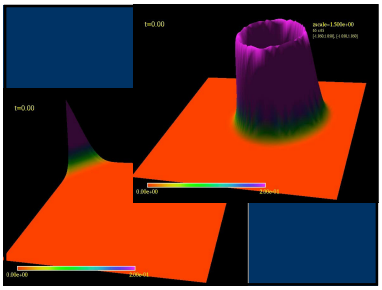
$$N = N_1 + iN_2 = \tan(\theta/2)e^{i\varphi}$$

- South patch similarly defined: $S = 1/N$

$$d\Omega^2 = \frac{d\theta^2 + \sin^2\theta d\varphi^2}{4\left(1 + N_1^2 + N_2^2\right)^{-2}\left[(dN_1)^2 + (dN_2)^2\right]}$$

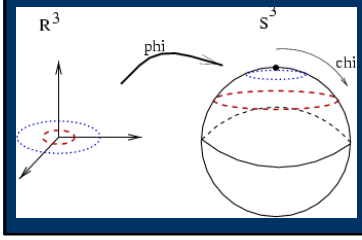
Other Models: Ricci Flow on Sphere

- Need to match up two patches
- Let patches overlap (extend past equator)
- Interpolate "boundary" values on one patch from values on other patch
- Define geometry on S^2 using two patches
- Test problem: Heat Equation
- Parabolic solves with an ADI solver



Long Term Goal: General Code

- Fully **gravitating**
- **3D** – no assumed spatial symmetry
- Coupled to **Magnetohydrodynamics (MHD)**
- Good general coordinates
- Resolved (far-field: waves; near-field: holes): **AMR**
- Robust – stable for strong field and matter sources
- Distributed for supercomputers/clusters – **MPI**
- Deals with singularities – **excision**



An Example…X-Y plane

- Show actual grid points
- Denser points equals fine grids