GOING BEYOND IDEAL MHD

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Overview

Maxwell equations in 3 regimes

- the ideal MHD limit
- the 3 regimes of Maxwell eqs.

Dealing with the stiff equation

- Approaches to the problem
- The IMEX Runge-Kutta methods

Application to the MHD equations

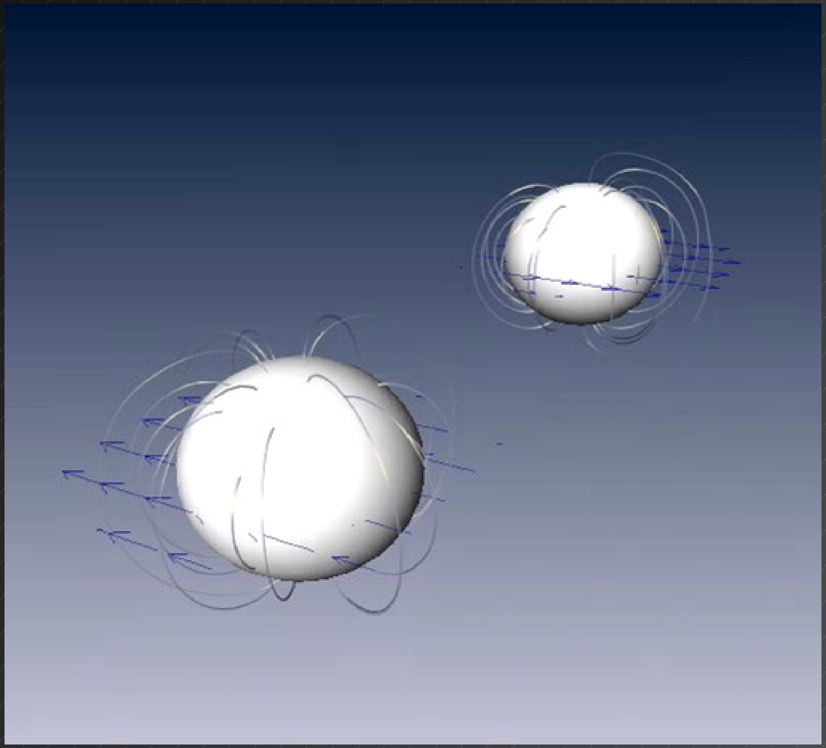
Summary and conclusions

Maxwell equations in 3 regimes

- The ideal MHD limit
- The 3 regimes of Maxwell eqs.

The ideal MHD limit

- Evolution of binary NS with magnetic fields



(talks of Bruno, Dave,…)

- initial (small ?) poloidal magnetic field

- the magnetic field is amplified by different mechanisms during the merger of the stars

- they can have an influence on the gravitational waves

The ideal MHD limit

- Maxwell equations for the EM fields (Komissarov talk)

$\nabla_a F^{ab} = -J^b$ $\quad$ F^{ab} : Faraday tensor

$\nabla_a {}^*F^{ab} = 0$ $\quad$ J^b : current 4-vector

- Einstein equations for the gravity (Alcubierre's talk)

$G_{ab} = 8\,\pi\, T_{ab}$ $\quad$ G_{ab} : Einstein tensor

- Hydrodynamic equations for the (ideal) fluid

$\nabla_a T^{ab} = 0$ $\quad$ T^{ab} : stress-energy tensor (ideal fluid + Maxwell)

$$T_{ab} = [\rho(1+\varepsilon) + P]u_a u_b + P\, g_{ab} + F_{ac} F^c{}_b - (F_{cd}F^{cd})g_{ab}/4$$

The ideal MHD limit

- Close the equations with a prescription for the current vector J^a

  * MHD approximation : time scale of the charged particles to reach the equilibrium is much shorter than other time scales in the problem

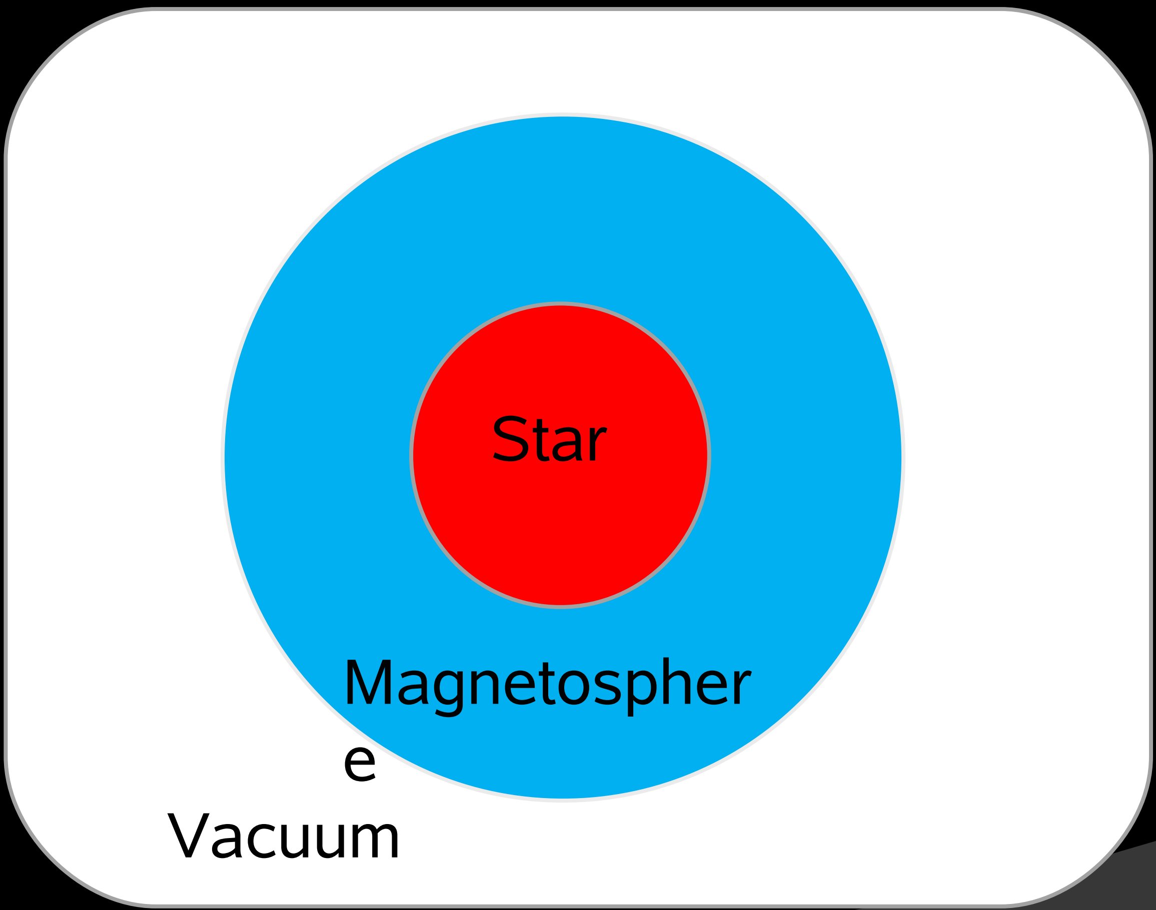
$$J_a + (J_b u^b)\, u_a = \sigma\, F_{ab}\, u^b$$

relativistic Ohm law σ :conductivity

  * Ideal MHD (no magnetic diffusion)

$$\sigma \rightarrow \infty \longrightarrow \begin{array}{l} \nabla_a {}^*F^{ab} = 0 \\ F_{ab}\, u^b = 0 \quad (\mathbf{E} = -\, \mathbf{v} \times \mathbf{B}) \end{array}$$

Maxwell equations in 3 regimes



- Star
  Perfect conductor

  IDEAL MHD

- Magnetosphere
  Dominated by the EM

  FORCE FREE

- Vacuum
  no sources

  MAXWELL EQS.

Maxwell equations in 3 regimes

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mathbf{J}$$
$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \cdot \mathbf{E} = q$$

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

σ : conductivity

IDEAL MHD
($\sigma \rightarrow \infty$)

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}$$
$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

FORCE FREE
($\mathbf{J} \times \mathbf{B} = 0$)

MAXWELL EQS.
($\sigma \rightarrow 0$)

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = 0$$
$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \cdot \mathbf{E} = 0$$

Maxwell equations in 3 regimes

- We are interested mainly on the dynamics in the star, but there are features involving the different regions.

- The jets and radiation are generated in the star, cross the magnetosphere and propagate on the vacuum

- A complete evolution of the different regions is necessary to describe jets and EM properly, but it is difficult to match solutions of different equations

- There can also be significant deviations from ideal MHD inside the star

- Naïve approach : evolve the full Maxwell equations with a generic Ohm's law in the three domains with no approximations, just changing σ

Hyperbolic-relaxation systems

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = q\,\mathbf{v} - \sigma W\,[\,\mathbf{v} \times \mathbf{B} + \mathbf{E} - (\mathbf{E}\cdot\mathbf{v})\,\mathbf{v}\,]$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$

$$\partial_t \mathbf{U} = F(\mathbf{U}) + R(\mathbf{U}) / \varepsilon$$

ε (= $1/\sigma$) : relaxation time

Hyperbolic-relaxation equation (STIFF)

difficult to evolve with standard numerical methods

Dealing with the stiff equation

- Approaches to the problem
- The IMEX Runge-Kutta methods

Approaches to the problem

- SOLUTION 1 : let us consider a simple case discretized with an explicit scheme

$$\partial_t \mathbf{U} = F(\mathbf{U}) + R(\mathbf{U}) / \varepsilon \quad \Leftrightarrow \quad \partial_t u = a\, \partial_x u - u / \varepsilon$$

$$(a=0): \quad u^{n+1} - u^n = - \Delta t\, u^n / \varepsilon \rightarrow \quad u^{n+1} = u^n (1- \Delta t/\varepsilon)$$

amplification factor $C^n = | u^{n+1}/u^n | < 1$ for stability

- CFL stability condition: $\Delta t < \Delta x / a$
- Stiff stability condition with explicit method: $\Delta t < 2\varepsilon$

if $\Delta t \sim \varepsilon = 1/\sigma \sim 10^{-6}$ → computationally impossible

Approaches to the problem

- SOLUTION 2 : solving the full equation implicitly

- Let us consider an implicit method

$$(a=0): \quad u^{n+1} - u^{n} = -\Delta t\, u^{n+1} / \varepsilon \;\rightarrow\; u^{n+1} = u^{n} / (1 + \Delta t/\varepsilon)$$

- Stiff stability condition with implicit method: $\Delta t > 0$
- But… it is expensive/complicated with non-vanishing F(U)

Approaches to the problem

- SOLUTION 3 : the equilibrium system
  - expand the solution around $\varepsilon \to 0$

$$\partial_t \mathbf{U} = F(\mathbf{U}) + R(\mathbf{U}) / \varepsilon \quad \Leftrightarrow \quad \partial_{tt} B - \Delta \mathbf{B} = [-\partial_t \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B})] / \varepsilon$$

$$\mathbf{U} = \mathbf{U}_0 + \varepsilon \, \mathbf{U}_1 + O(\varepsilon^2) \qquad \mathbf{B} = \mathbf{B}_0 + \varepsilon \, \mathbf{B}_1 + O(\varepsilon^2)$$

$O(\varepsilon^0)$: IDEAL MHD $\qquad \partial_t \mathbf{B_0} - \nabla \times (\mathbf{v} \times \mathbf{B_0}) = 0$

$O(\varepsilon^1)$: $\qquad \partial_t \mathbf{B_1} - \nabla \times (\mathbf{v} \times \mathbf{B_1}) = - (\partial_{tt} \mathbf{B}_0 - \Delta \, \mathbf{B}_0)$

- hierarchy of solutions : compute $\mathbf{B}_0$, then $\mathbf{B}_1$,… but it is only valid close to $\varepsilon \to 0$

Approaches to the problem

- SOLUTION 4 : Strang Splitting

$\partial_t \mathbf{U} = F(\mathbf{U}) + R(\mathbf{U}) / \varepsilon$ ⟺ $\partial_t \mathbf{U} = S(\Delta t/2)$ ¤ $T(\Delta t)$ ¤ $S(\Delta t/2)\, \mathbf{U}$

U^{*} : $U^{*} = U^{n} + (\Delta t/2)\, R(U^{n}) /\varepsilon$

U^{**} : $U^{**} = U^{*} + \Delta t\, F(U^{*})$

U^{n+1}: $U^{n+1} = U^{**} + (\Delta t/2)\, R(U^{**})/\varepsilon$

- The source step can be solved exactly with the analytical solution (Komissarov 2007)… but it does not work for strong stiff terms in the presence of shocks

The IMEX Runge Kutta methods

- treat implicitly the stiff part and explicitly the non-stiff (IMplicit-EXplicit methods)

$$\partial_t \mathbf{U} = F(\mathbf{U}) + R(\mathbf{U}) / \varepsilon$$

$$U^{(i)} = U^n + \Delta t \, \Sigma \, \underline{a}_{ij} \, F(U^{(j)}) + \Delta t \, \Sigma \, a_{ij} \, R(U^{(j)}) / \varepsilon$$

$$U^{n+1} = U^n + \Delta t \, \Sigma \, \underline{\omega}_i \, F(U^{(i)}) + \Delta t \, \Sigma \, \omega_i \, R(U^{(i)}) / \varepsilon$$

$$\begin{array}{c|cccc} \underline{c}_1 & 0 & 0 & 0 & \dots \\ \underline{c}_2 & \underline{a}_{12} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \underline{c}_n & \underline{a}_{1n} & \underline{a}_{2n} & \dots & 0 \\ \hline & \underline{\omega}_1 & \underline{\omega}_2 & \dots & \underline{\omega}_n \end{array}$$

Explicit RK

DIRK

$$\begin{array}{c|cccc} c_1 & a_{11} & 0 & 0 & \dots \\ c_2 & a_{12} & a_{22} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ c_n & a_{1n} & a_{2n} & \dots & a_{nn} \\ \hline & \omega_1 & \omega_2 & \dots & \omega_n \end{array}$$

Butcher Tableau

The IMEX Runge Kutta methods

- Let us consider a simple IMEX RK as an example

$$\partial_t \mathbf{U} = F(\mathbf{U}) + R(\mathbf{U}) / \varepsilon$$

IMEX-Midpoint(1,2,2)

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ \hline & 0 & 1 \end{array}$$

$$\mathbf{U}^1 = \mathbf{U}^n$$

$$\mathbf{U}^2 = \mathbf{U}^n + \Delta t\, F(\mathbf{U}^1)/2 + \Delta t\, R(\mathbf{U}^2)/(2\,\varepsilon)$$

$$U^{n+1} = U^n + \Delta t\, F(U^2) + \Delta t\, R(U^2) / \varepsilon$$

- only the stiff part has to be inverted
- high order convergence in time (usually 3 order)
- strong theoretical background (it has to work!)

Application to the Maxwell equations

- Inverting explicitly the stiff part
- Numerical tests

Inverting explicitely the stiff part

- only the evolution of the electric field has stiff terms

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = q\,\mathbf{v} - \sigma\, W\, [\, \mathbf{v} \times \mathbf{B} + \mathbf{E} - (\mathbf{E}\cdot\mathbf{v})\, \mathbf{v}]$$

- use standard TVD RK methods for the other fields and apply the IMEX only to E

$$\partial_t \mathbf{U} = F(\mathbf{U}) + R(\mathbf{U}) / \varepsilon \quad \longrightarrow \quad \begin{aligned} F(\mathbf{E}) &= \nabla \times \mathbf{B} + q\,\mathbf{v} \\ R(\mathbf{E}) &= -\,W\, [\mathbf{v} \times \mathbf{B} + \mathbf{E} - (\mathbf{E}\cdot\mathbf{v})\, \mathbf{v}] \\ S &= -\,W\, \mathbf{v} \times \mathbf{B} \end{aligned}$$

Inverting explicitly the stiff part

Example:

$$\mathbf{U}^1 = \mathbf{U}^n$$

$$\mathbf{U}^2 = \mathbf{U}^n + \Delta t\, F(\mathbf{U}^1)/2 + \Delta t\, R(\mathbf{U}^2)/(2\,\varepsilon)$$

$$U^{n+1} = U^n + \Delta t\, F(U^2) + \Delta t\, R(U^2)/\varepsilon$$

- compute the explicit part

$$\mathbf{E}^* = \mathbf{E}^n + \Delta t\, F(\mathbf{E}^1)/2$$

- invert explicitly the implicit part

$$E^2 = M(v, B)\,[\,E^* + \Delta t\, S/(2\,\varepsilon)\,]$$

- compute $F(E^2)$ and $R(E^2)$ to update E^{n+1}

Test 1: the Alfven wave (del Zanna 2007)

• Testing the high co

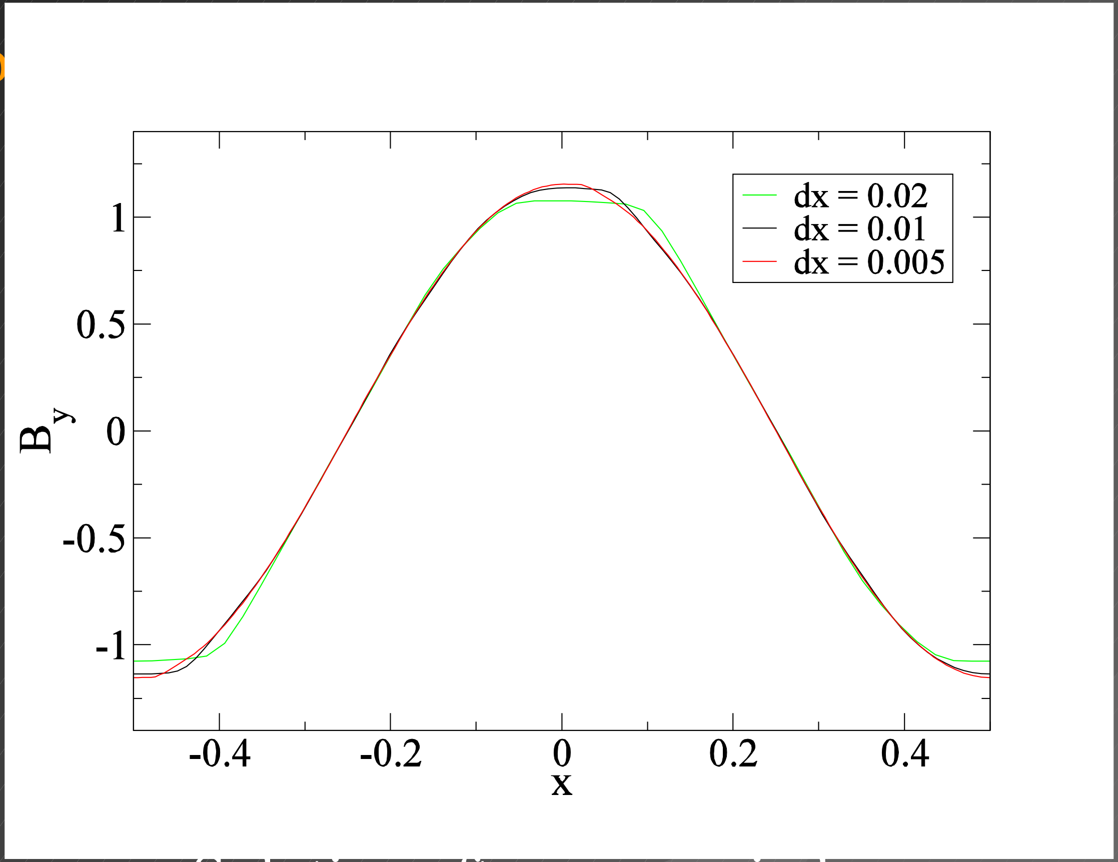
$B_y = B_o \cos(x - v_A t)$

$B_z = B_o \sin(x - v_A t)$

$v_y = -v_A B_y/B_o$

$v_z = -v_A B_z/B_o$

Alfven speed v_A



Solution after one period
(periodic boundary conditions)

$P=\rho=1$, $v_A=1/2$

conductivity $\sigma = 10^6$

Test 2: the current sheet (Komissarov 2007)

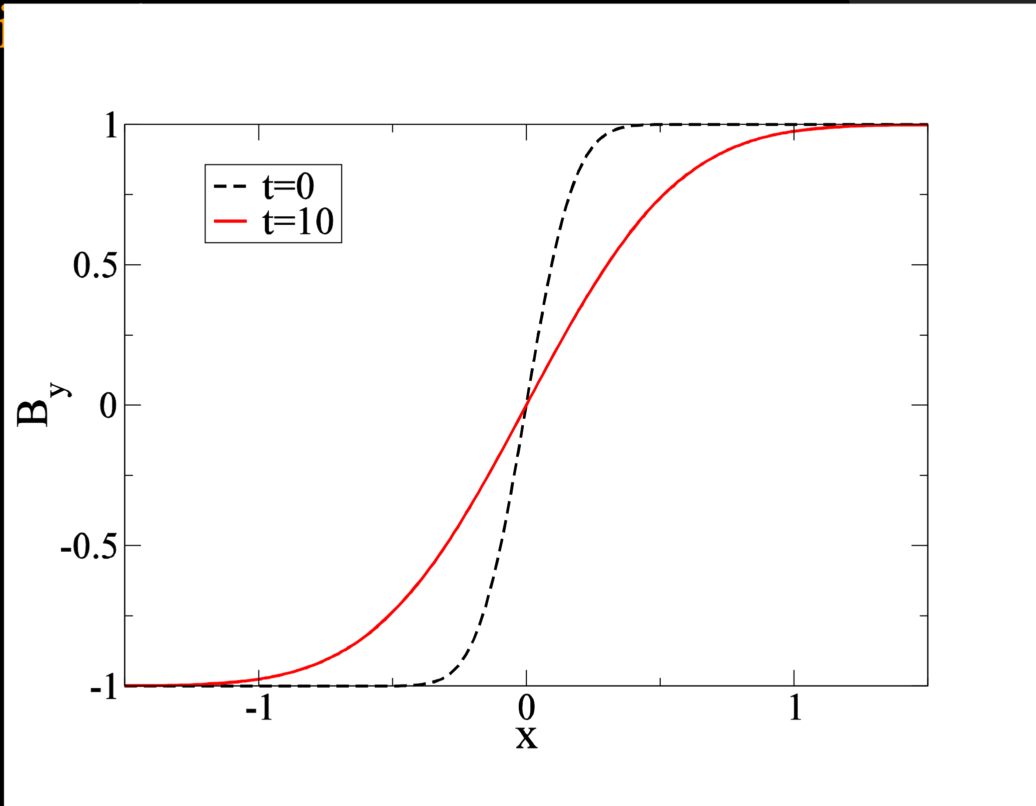
- Testing the low conducti

P=cte, ρ=cte
$E = v = 0$
$B= (0,B_y(x,t),0)$

$$\partial_t B_y - (1/\sigma)\, \partial_{xx} B_y = 0$$

$$B_y = B_o \,\mathrm{erf}[(\sigma/(4\,\xi))^{1/2}]$$

with $\xi=t/x^2$

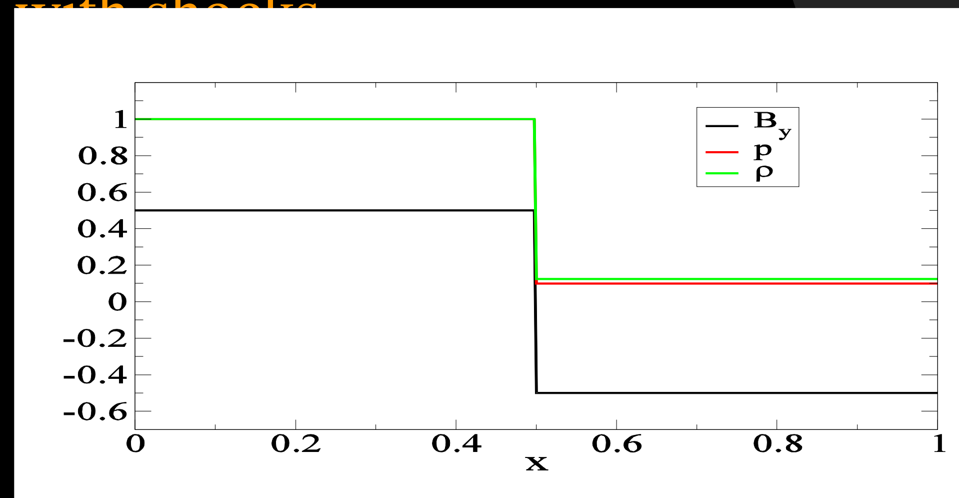


Solution at t=10 with σ=100

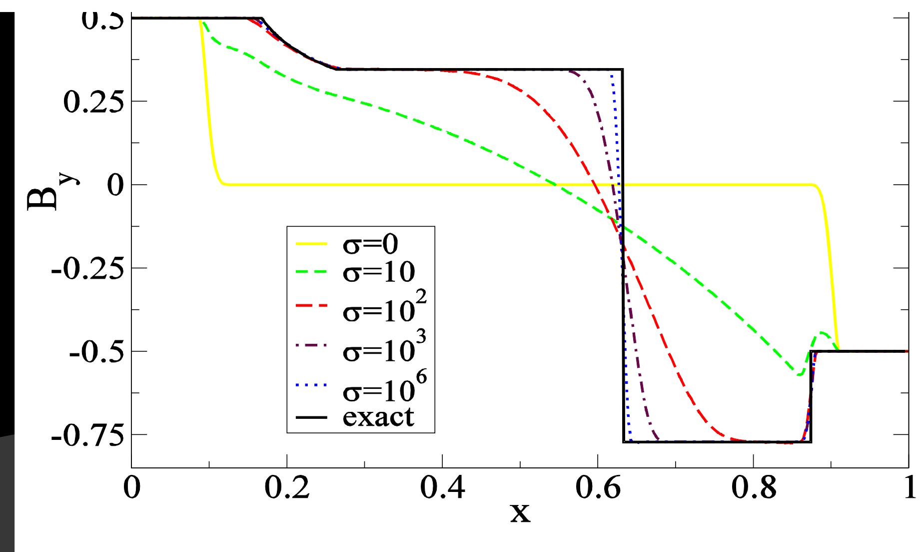
Test 3: the shock tube problem

- Testing the resistive MHD with shocks

Left state
$(\rho^L, p^L, B_y{}^L) = (1, 1, 1/2)$
Right state
$(\rho^R, p^R, B_y{}^R) = (1/8, 0.1, -1/2)$



Solution at t=0.4



Test 4: the cylindrical explosion

- Testing the resistive MHD with shocks in 2D

r<0.8 p=1, ρ = 0.01

r>1.0 p= ρ = 0.001

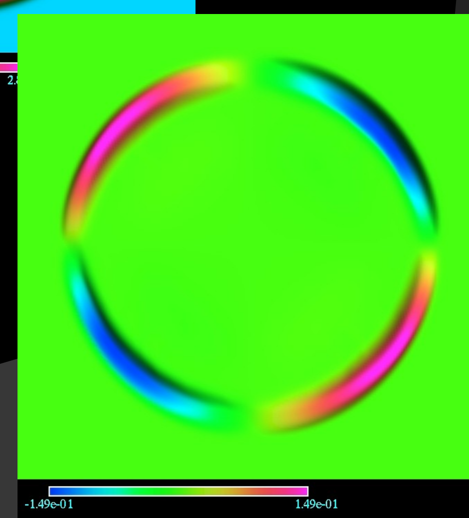
B = (0.05, 0, 0)

E = q = 0

B_x

Solution at t=4

B_y



Test 5: the cylindrical star

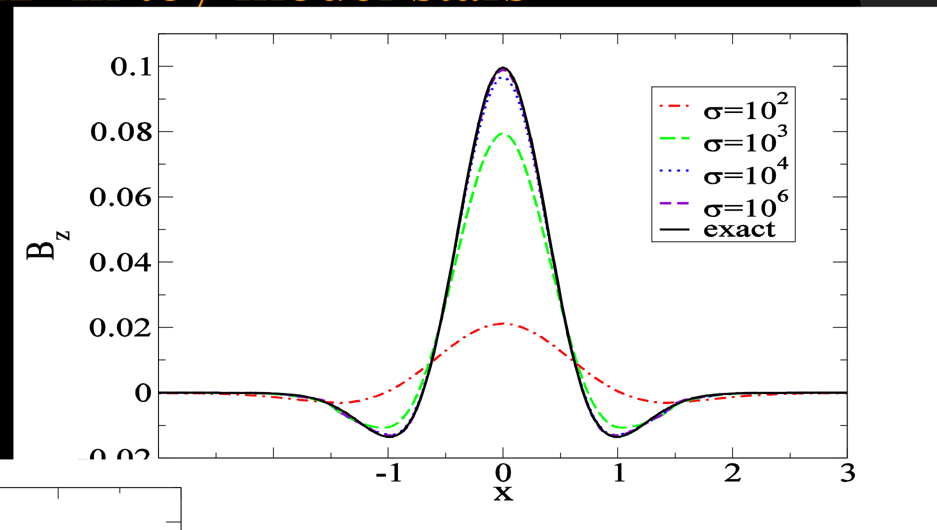
- Testing the resistive MHD in toy model stars

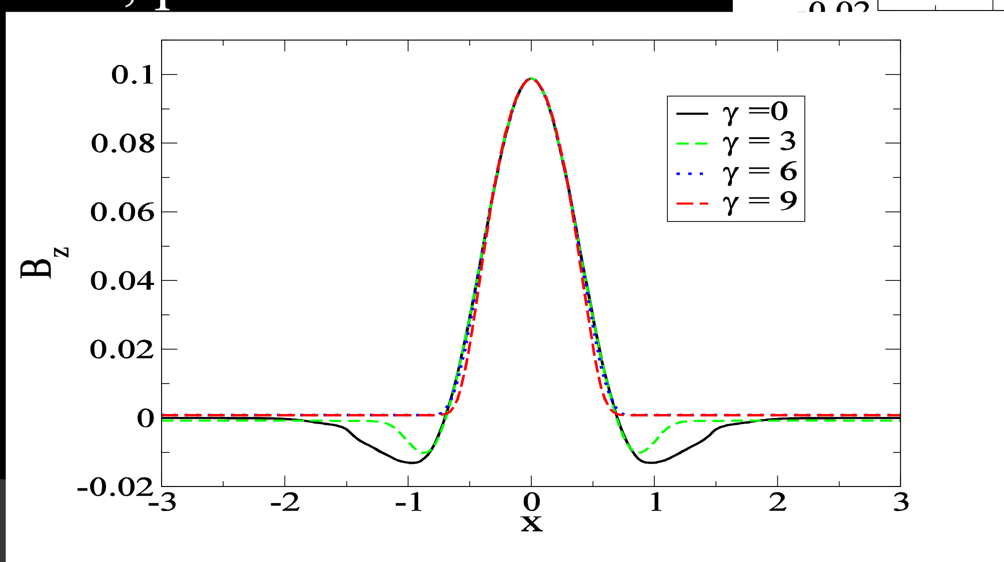
$\rho = \rho_0 \exp[-(r/r_o)^2]$

$v_\varphi = \rho\,\Omega$

$B_z = 2\,B_o[1 - (r/r_o)^2]$

E,q from ideal MHD





$\sigma = \sigma_0\, D^\gamma$ $D = \rho W$

Summary and conclusions

• we have introduced the IMEX Runge-Kutta to solve the resistive MHD equations

• the IMEX has been implemented successfully in a simple fully explicit scheme and tested with several tests

• But not everything is good…
  - the inversion from conserved to primitive is slow and more delicate than in the ideal MHD

Future work

- Improve the robustness and speed of the conversion from conserved to primitive

- Generalize the Ohm law to include some sort of force-free limit in the magnetosphere

- Generalize the MHD eqs. to full GR

- Compare the binary NS merger with ideal MHD versus resistive MHD